THE SPECTRA OF OPERATORS HAVING RESOLVENTS OF FIRST-ORDER GROWTH(1)

BY

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1. **Introduction.** Throughout this paper, $\mathfrak{H}$ will denote an infinite-dimensional Hilbert space of vectors $x, y, \ldots$, with inner product (x, y), and T will denote a bounded operator on $\mathfrak{H}$ with spectrum sp (T). As usual, let $\|T\| = \sup \|Tx\|$ where $\|x\| = 1$. If λ does not belong to sp (T), put $R_\lambda = (T-\lambda I)^{-1}$, and let $d(\lambda)$ denote the distance from λ to sp (T), thus,

$$d(\lambda) = \inf |\lambda-\mu|, \quad \text{where } \mu \in \text{sp}\,(T). \tag{1.1}$$

There will be studied certain properties of points λ_0 in the boundary of sp (T) when the resolvent R_λ satisfies the growth condition

$$\begin{gathered} d(\lambda)\|R_\lambda\| \to 1 \quad \text{as } \lambda \to \mu \\ \text{for all } \mu \text{ in the boundary of sp}\,(T) \text{ and in some neighborhood of } \lambda_0. \end{gathered} \tag{1.2}$$

It is well known and easy to show that for any T one has $d(\lambda)\|R_\lambda\| \geqq 1$ for all $\lambda \notin$ sp (T) and that $d(\lambda)\|R_\lambda\| \to 1$ as $|\lambda| \to \infty$. The extreme possibility

$$d(\lambda)\|R_\lambda\| = 1 \quad \text{for } \lambda \notin \text{sp}\,(T), \tag{1.3}$$

which of course implies (1.2), certainly holds for normal operators as well as for others which are "nearly" normal; for instance, it is satisfied by seminormal operators T, so that

$$T^*T-TT^* \text{ is semidefinite.} \tag{1.4}$$

See Stampfli [11], also the references given there to Donoghue and Nieminen.

Recall that a sequence $\{x_n\}$ of vectors is said to converge weakly to a limit x as $n \to \infty$ (notation $w: x_n \to x$) if $(x_n, y) \to (x, y)$ as $n \to \infty$ for all y in $\mathfrak{H}$. It will be convenient to define for any bounded operator T the sets $A(T)$ and $B(T)$ by

$$\begin{aligned} A(T) = [\lambda: T_\lambda x_n \to 0,\ T_\lambda^* y_n \to 0 \text{ for some pair of sequences } \{x_n\}, \{y_n\} \\ \text{satisfying } \|x_n\| = \|y_n\| = 1 \text{ and } w: x_n \to 0,\ w: y_n \to 0 \text{ as } n \to \infty] \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} B(T) = [\lambda: T_\lambda x_n \to 0,\ T_\lambda^* x_n \to 0 \text{ for some sequence } \{x_n\} \text{ satisfying} \\ \|x_n\| = 1 \text{ and } w: x_n \to 0 \text{ as } n \to \infty]. \end{aligned} \tag{1.6}$$

(Here and in the sequel, $T_\lambda = T-\lambda I$.)

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Let $W(T)$ be the set of points of sp (T) which are invariant under all completely continuous perturbations of T, thus, $W(T)=\bigcap_C \operatorname{sp}(T+C)$, where C is completely continuous. (For a characterization of $W(T)$, see Schechter [9].) It is clear that

$$B(T) \subset A(T) \subset W(T). \tag{1.7}$$

For any T, let $E(T)$ denote the set of limit (cluster) points of sp (T) together with all points of the point spectrum of T of infinite multiplicity. Then, in case T is normal, $E(T)$ constitutes the essential spectrum of T and, according to Weyl's theorem (Weyl [12]), $E(T)=B(T)$.

It is easy to see that if T is normal, $E(T)=W(T)$. It was shown by Coburn [2], using properties of $W(T)$ given in Schechter [9], that this last equation remains valid even for seminormal operators. The corresponding assertion, even for the direct sum of two seminormal operators, is false however. In fact, if V denotes the isometric operator on the sequential Hilbert space $\mathfrak{K}$ of vectors $x=(a_1, a_2, \ldots)$ defined by $V\colon (a_1, a_2, \ldots) \to (0, a_1, a_2, \ldots)$, then V and V^* are seminormal, each has as its spectrum the unit disk $|\lambda| \leqq 1$ on $\mathfrak{K}$ and hence so also does $T_0=V \oplus V^*$ on $\mathfrak{H}=\mathfrak{K} \oplus \mathfrak{K}$, and, in particular, $E(T_0)=\operatorname{sp}(T_0)=\{\lambda : |\lambda| \leqq 1\}$. Although T_0 is not seminormal, it does satisfy (1.3), since V and V^* do separately. But there exist completely continuous operators C, even with C arbitrarily small, on $\mathfrak{H}=\mathfrak{K} \oplus \mathfrak{K}$, such that sp $(T+C)$ is the boundary, $|\lambda|=1$, of sp (T_0); see Putnam [7] (also Halmos [5, Solutions 85, 144]). Thus $W(T_0)$ is a subset of the set $|\lambda|=1$. It follows from Theorem 2 below however that the set $|\lambda|=1$ is contained in $B(T_0)$ and, since $B(T_0) \subset W(T_0)$, $W(T_0)$ is precisely the set $|\lambda|=1$.

There will be proved the following theorems:

THEOREM 1. *Let T be a bounded operator and let λ_0 be a nonisolated point of the boundary of* sp (T). *Then λ_0 belongs to the set $A(T)$ of* (1.5).

THEOREM 2. *Let T be a bounded operator and let λ_0 be a nonisolated point of the boundary of* sp (T) *for which* (1.2) *holds. Then λ_0 belongs to the set $B(T)$ of* (1.6).

It follows from Theorem 1 and (1.7) that $W(T)$ always contains the set of nonisolated boundary points of sp (T). That $W(T)$ may coincide with this latter set, even if (1.3) holds, is seen from the example mentioned above.

In §4, some other spectral implications of Theorems 1 and 2 will be derived.

REMARKS. The author is indebted to M. Schechter for the present formulation of Theorem 1 and its proof below. The author's original version involved an added hypothesis on the growth of the resolvent as well as a considerably longer proof.

2. **Proof of Theorem 1.** Since λ_0 is a nonisolated boundary point of sp (T) if and only if $\bar{\lambda}_0$ is a nonisolated boundary point of sp (T^*), it is clearly sufficient to prove the existence of the sequence $\{x_n\}$, for $\lambda=\lambda_0$, in (1.5). Also, according to Wolf [13, p. 215], the existence of such a sequence is equivalent to the statement that either $\mathfrak{R}(T_{\lambda_0})$ is not closed or that the dimension $\alpha(T_{\lambda_0})$ of the null space of T_{λ_0} is

infinite. The proof will be completed then by showing that the assumptions that $\mathfrak{R}(T_{\lambda_0})$ be closed and that $\alpha(T_{\lambda_0})$ be finite lead to a contradiction.

To this end, note that $\mathfrak{R}(T_\lambda)$ is closed and $\alpha(T_\lambda)$ is finite for all λ sufficiently close to λ_0; see Wolf [13, p. 216]. Also, the range of a bounded operator is closed if and only if the range of its adjoint is also closed; cf. Goldberg [4, p. 95]. Since $\alpha(T_{\lambda_0})$ and $\alpha(T^*_{\lambda_0})$ are not both infinite, so that T_{λ_0} has an index, then $\alpha(T_\lambda)$ and $\alpha(T^*_\lambda)$ are constant on a punctured neighborhood of λ_0. This follows from results of Gohberg and Kreĭn [3]; see Goldberg [4, p. 114], wherein can be found a generalization. Since λ_0 is a boundary point of the resolvent set of T it follows that for some $\delta>0$, $\alpha(T_\lambda)=\alpha(T^*_\lambda)=0$, and both $\mathfrak{R}(T_\lambda)$ and $\mathfrak{R}(T^*_\lambda)$ are closed for $0<|\lambda-\lambda_0|<\delta$. Thus the set $\{\lambda : 0<|\lambda-\lambda_0|<\delta\}$ belongs to the resolvent set of T, a contradiction.

3. **Proof of Theorem 2.** For use below it will be convenient to have the following

LEMMA. *Let $\lambda_n \to \lambda_0$ as $n\to\infty$ and suppose that for each n, there exists a sequence $\{x_{nk}\}$, $k=1, 2, \ldots$, of unit vectors satisfying $w: x_{nk}\to 0$ and $T_{\lambda_n}x_{nk}\to 0$ as $k\to\infty$. Then there exists a sequence $\{x_n\}$ of the form $x_n=x_{nk_n}$ $(k_1<k_2<\cdots)$ satisfying*

$$\|x_n\| = 1, \quad w: x_n \to 0 \quad \textit{and} \quad T_{\lambda_0}x_n \to 0 \quad \textit{as } n\to\infty. \tag{3.1}$$

Proof. Let $\mathfrak{M}$ denote the (separable) space spanned by $\{x_{nk}\}$ for $n, k=1, 2, \ldots$, and let $\mathfrak{M}=\{\phi_k\}$ be a complete (countable) orthonormal system spanning $\mathfrak{M}$. It is clear that one can choose $k=k_n$ satisfying $k_1<k_2<\cdots$ and so that $|(x_{nk_n}, \phi_j)|<1/n$ for $j=1, 2, \ldots, n$ and $\|T_{\lambda_n}x_{nk_n}\|<1/n$. If x is any vector in $\mathfrak{H}$, then $x=y+z$, where $y\in\mathfrak{M}$ and $z\in\mathfrak{M}^\perp$. Then $(x, x_{nk_n})=(y, x_{nk_n})\to 0$ as $n\to\infty$, and it is clear that the sequence $\{x_n\}$ defined by $x_n=x_{nk_n}$ satisfies the required conditions. This completes the proof of the Lemma.

Next, let an eigenvalue λ of any operator T be called a normal eigenvalue if $\bar\lambda$ is an eigenvalue of T^* and if the null spaces of T_λ and T^*_λ coincide. It was proved by Stampfli [11] that if T satisfies (1.3) then any isolated point of sp (T) must be a normal eigenvalue. (The special case in which T satisfies (1.4) was treated in Stampfli [10].) The same argument shows that if λ_0 is an isolated point of sp (T) and if $d(\lambda)\|R_\lambda\|\to 1$ as $\lambda\to\lambda_0$ then λ_0 is a normal eigenvalue.

In order to prove Theorem 2, note that if there exists a sequence $\{\lambda_n\}$ of distinct isolated points λ_n in sp (T) satisfying $\lambda_n\to\lambda_0$ as $n\to\infty$ then the λ_n must be normal eigenvalues. Hence $(x_n, x_m)=0$ if $T_{\lambda_n}x_n=0$ and $T_{\lambda_m}x_m=0$ and $n\neq m$. Thus $\{x_n\}$ is an infinite orthonormal sequence satisfying $T_{\lambda_0}x_n\to 0$ and $T^*_{\lambda_0}x_n\to 0$ (and, of course, $w: x_n\to 0$), as $n\to\infty$, so that the assertion of Theorem 2 follows.

If λ_0 is not the limit of a sequence of isolated points of sp (T) then it is easy to see that there exists a sequence $\{\lambda_n\}$ of distinct points satisfying $\lambda_n\to\lambda_0$ as $n\to\infty$, such that each λ_n is a nonisolated boundary point of sp (T) and, for each λ_n, there is a disk C_n containing λ_n on its boundary and having an interior free of points of sp (T). By Theorem 1, for each λ_n, there exists a sequence $\{x_{nk}\}$, $k=1, 2, \ldots$, of unit vectors satisfying $w: x_{nk} \to 0$ and $T_{\lambda_n} x_{nk} \to 0$ as $k \to \infty$. It will next be shown that also $T^*_{\lambda_n} x_{nk} \to 0$ as $k \to \infty$.

For $\lambda \notin$ sp (T), $T_\lambda x_{nk} = T_{\lambda_n} x_{nk} + (\lambda_n - \lambda) x_{nk}$ and hence, on multiplying by R_λ,

$$((\lambda-\lambda_n)R_\lambda + I)x_{nk} \to 0 \quad \text{as } k \to \infty, \tag{3.2}$$

for λ, n fixed. But $0 \leqq \|((\bar{\lambda}-\bar{\lambda}_n)R^*_\lambda + I)x_{nk}\|^2 = |\lambda-\lambda_n|^2 \|R^*_\lambda x_{nk}\|^2 + (x_{nk}, (\lambda-\lambda_n)R_\lambda x_{nk}) + ((\lambda-\lambda_n)R_\lambda x_{nk}, x_{nk}) + \|x_{nk}\|^2$. In view of (3.2) this implies that

$$\|((\bar{\lambda}-\bar{\lambda}_n)R^*_\lambda + I)x_{nk}\|^2 \leqq |\lambda-\lambda_n|^2 \|R^*_\lambda x_{nk}\|^2 - 1 + \eta_k, \tag{3.3}$$

where $\eta_k \to 0$ as $k \to \infty$ (for λ, n fixed).

Now, let r_n denote that radius of C_n containing the point λ_n. Since $d(\lambda) = |\lambda - \lambda_n|$ for λ on r_n, it follows from (1.2) and (3.3) that for each $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ and a positive integer N_ε with the property that $\|((\bar{\lambda}-\bar{\lambda}_n)R^*_\lambda + I)x_{nk}\| < \varepsilon$ provided λ (fixed) is on r_n and $0 < |\lambda - \lambda_n| < \delta_\varepsilon$, $k > N_\varepsilon$. Since $T^*_\lambda((\bar{\lambda}-\bar{\lambda}_n)R^*_\lambda + I) = T^*_{\lambda_n}$, it is clear that $T^*_{\lambda_n} x_{nk} \to 0$ as $k \to \infty$.

Consequently, it has been shown that there exist $\lambda_n \to \lambda_0$ and sequences $\{x_{nk}\}$ of unit vectors satisfying $w: x_{nk} \to 0$ and both limit relations $T_{\lambda_n} x_{nk} \to 0$ and $T^*_{\lambda_n} x_{nk} \to 0$ as $k \to \infty$. An application of the Lemma then yields the existence of a sequence $\{x_n\}$ of unit vectors satisfying $w: x_n \to 0$ and for which both $T_{\lambda_0} x_n \to 0$ and $T^*_{\lambda_0} x_n \to 0$ as $n \to \infty$. Thus λ_0 is in the set $B(T)$ of (1.6) and the proof of Theorem 2 is now complete.

4. **Some spectral properties.** Let T be seminormal or, more generally, satisfy (1.3), or even

$$d(\lambda)\|R_\lambda\| \to 1 \quad \text{as } \lambda \to \lambda_0 \quad (\lambda \notin \text{sp}(T)), \tag{4.1}$$

for all λ_0 in the boundary of sp (T). If $C = T^*T - TT^*$, it is clear that

$$C = T^*_\lambda T_\lambda - T_\lambda T^*_\lambda \quad \text{for arbitrary } \lambda. \tag{4.2}$$

Hence, as a consequence of Theorem 2, if the boundary of sp (T) contains at least one nonisolated point (that is, if sp (T) is an infinite set), there exists a sequence $\{x_n\}$ of unit vectors satisfying $w: x_n \to 0$ and $Cx_n \to 0$ as $n \to \infty$. This result holds also if sp (T) contains only a finite number of points, for, as noted earlier, each such point must be a normal eigenvalue, and one of these, therefore, must be of infinite multiplicity (the Hilbert space being infinite dimensional). Hence $\{x_n\}$ can be chosen to be an orthonormal sequence of eigenvectors corresponding to this eigenvalue. Thus, there has been proved the following:

THEOREM 3. *If T is a bounded operator satisfying* (4.1) *for all λ_0 in the boundary of* sp (T), *then* 0 *belongs to the essential spectrum of $T^*T - TT^*$.*

REMARKS. It is seen that if, in particular, T is seminormal then it is necessary that 0 belong to the essential spectrum of $T^*T - TT^*$. It is easy to see also that if S is any compact set of nonnegative (or nonpositive) real numbers containing 0, then there exists a seminormal operator T for which sp $(T^*T-TT^*)=S$. In fact, if V is the isometric operator considered in §1, it is seen that $D=V^*V-VV^*$ is the diagonal matrix all elements of which are 0 except for 1 in the (1, 1) position. In particular, sp (D) consists of 0 and 1. If $\{r_n\}$ is any countable set of nonnegative real numbers whose closure is S it is seen that for the direct sum operator $T=\bigoplus_{n=1}^{\infty} r_n^{1/2}V$ on the space $\mathfrak{H}=\bigoplus_{n=1}^{\infty}\mathfrak{K}$, one has sp $(T^*T-TT^*)=S$.

Added in proof. In fact, it follows from a result of H. Radjavi (J. Math. Mech. **16** (1966), 19–26) that a nonnegative operator C on an infinite-dimensional separable Hilbert space has 0 in its essential spectrum if and only if it is of the form $C=T^*T-TT^*$.

Another consequence of Theorem 2 is the following:

THEOREM 4. *Let T be a bounded operator satisfying* (4.1) *for all λ_0 in the boundary of* sp (T) *and let T have the Cartesian representation $T=H+iJ$, where H and J are selfadjoint. Then the projections of* sp (T) *onto the x and y axes are, respectively, contained in the sets* sp (H) *and* sp (J).

In order to see this, it is sufficient to consider the operator $H\ (=\frac{1}{2}(T+T^*))$ only. Let $\lambda_0 \in$ sp (T). If λ_0 is an isolated point of sp (T), it is a normal eigenvalue (cf. §3 above) and clearly Re (λ_0) is in sp (H). If λ_0 is not an isolated point of sp (T) then it is clear that there exists some nonisolated boundary point λ_1 of sp (T) satisfying Re $(\lambda_1)=$ Re (λ_0). Hence, by Theorem 2, Re (λ_0) is in sp (H), in fact, Re (λ_0) is in the essential spectrum of H.

REMARK. In case T is seminormal, that is, if (1.4) holds, the sets sp (H) and sp (J) are precisely the projections of sp (T) onto the x and y axes (Putnam [6]). Whether the corresponding assertion holds if the condition of seminormality is relaxed to (4.1) for all λ_0 in the boundary of sp (T), or even to (1.3), is apparently not known.

An assertion related to that of Theorem 4 can be made for any operator T for which $C=T^*T-TT^*$ is completely continuous. For, by (4.2), it is clear that $A(T)=B(T)$. It follows from Theorem 1 that the projections of all nonisolated boundary points of sp (T) onto the x and y axes belong to the spectra (even essential spectra) of H and J respectively.

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